contradiction. From Eqs. (6) and (13), respectively, it is seen that the gage function of the leading term in the outer expansion would be $v^2$ and that the gage functions of the terms $\alpha^2\phi_2$, $\alpha^3\phi_3$, and $\alpha^4\phi_4$, of the inner expansion expressed in outer variables would be greater than $v^2$.

A direct comparison of the present results and those of Ref. 3 can be made for the intermediate case. It can be seen from Eq. (29) that the radial derivative of $\varphi$ can be written in outer variables near the limit $\bar{r} \to 0$ as

$$\frac{2\pi r}{U_\infty l \delta^2}\frac{\partial \varphi}{\partial r} = -\overset{\text{doublet}}{\frac{2\pi}{\mu}f(\bar{x})\frac{\sin\omega}{\bar{r}}} + \overset{\text{source}}{S'(\bar{x})} + \overset{\text{source}}{\frac{(\gamma+1)\pi}{\mu}f'(\bar{x})f''(\bar{x})} \qquad (30)$$

The right side of this equation can be compared with the right side of Eq. (19) in Ref. 3. It can be seen that, in addition to the doublet due to lift and source due to thickness obtained in Ref. 3, the present results show a source which depends on the lift distribution function $f(\bar{x})$ (and hence the planform shape).


**References**

[1] Heaslet, M. A. and Spreiter, J. R., "Three-Dimensional Transonic Flow Theory Applied to Slender Wings and Bodies," Rept. 1318, 1957, NACA.

[2] Hayes, W. D., "The Second-Order Approximation for Transonic Nonviscous Flow," *Journal de Mécanique*, Vol. 5, No. 2, June 1966, pp. 163–206.

[3] Cheng, H. K. and Hafez, M., "Three-Dimensional Structure and Equivalence Rule of Transonic Flows," *AIAA Journal*, Vol. 10, No. 8, Aug. 1972, pp. 1115–1117.

[4] Cole, J. D. and Messiter, A. F., "Expansion Procedures and Similarity Laws for Transonic Flow, Part 1. Slender Bodies at Zero Incidence," *Zeitschrift für Angewandte Mathematik und Physik*, Vol. 8, 1957, pp. 1–25.


# Technical Comments

## Comment on "Discrete Element Idealization of an Incompressible Liquid for Vibration Analysis" and "Discrete Element Structural Theory of Fluids"


ROBERT D. COOK*
*University of Wisconsin, Madison, Wis.*


### Nomenclature

- $\mathbf{f}$ = vector of $x$, $y$, $z$ displacements, $\mathbf{f} = \{uvw\}$
- $\mathbf{q}$ = vector of $x$, $y$, $z$ nodal point displacements
- $\boldsymbol{\sigma}$ = engineering stresses, $\boldsymbol{\sigma} = \{\sigma_x \sigma_y \sigma_z \tau_{xy} \tau_{yz} \tau_{zx}\}$
- $\boldsymbol{\varepsilon}$ = engineering strains, $\boldsymbol{\varepsilon} = \{\varepsilon_x \varepsilon_y \varepsilon_z \gamma_{xy} \gamma_{yz} \gamma_{zx}\}$
- $\varepsilon_V$ = volumetric strain, $\varepsilon_V = \varepsilon_x + \varepsilon_y + \varepsilon_z$
- $\mathbf{N}$ = shape function matrix, $\mathbf{f} = \mathbf{Nq}$
- $\mathbf{N}_3$ = third row of $\mathbf{N}$, $w = \mathbf{N}_3\mathbf{q}$
- $\mathbf{B}$ = strain-displacement matrix, $\boldsymbol{\varepsilon} = \mathbf{Bq}$
- $\mathbf{D}$ = volumetric strain-displacement matrix, $\varepsilon_V = \mathbf{Dq}$, $\mathbf{D} = \mathbf{B}_1 + \mathbf{B}_2 + \mathbf{B}_3$
- $\mathbf{C}$ = stress-strain rate matrix, $\boldsymbol{\sigma} = \mathbf{C}\dot{\boldsymbol{\varepsilon}}$
- $K$ = bulk modulus
- $\mu$ = dynamic coefficient of viscosity
- $\rho$ = mass density of fluid
- $g$ = acceleration of gravity (down is $-z$ direction)
- $S$ = surface tension of fluid
- $L$ = Lagrangian, kinetic minus potential energy
- $\partial V$ = boundary of fluid element of volume $V$
- $\hat{\mathbf{k}}$ = unit vector in upward ($+z$) direction
- $\hat{\mathbf{n}}$ = outwardly directed unit normal to element
- $\omega$ = circular frequency of sloshing, rad/sec

TWO recent papers[1,2] discuss a "structural theory" of fluids. A finite element solution which resembles that used in solid mechanics is developed, using physical reasoning and intuition to account for the effects of sloshing, surface tension, etc. In the present Comment this formulation is placed on a theoretical foundation, thus providing further insight and opening the way toward use of a variety of finite element models. The following is expressed in the usual finite-element notation[3] and presumes use of assumed displacement fields.


Received August 25, 1972.

Index categories: Wave Motion and Slashing; Structural Dynamic Analysis.



* Associate Professor, Department of Engineering Mechanics.


The development is based on Lagrange's equations, which are[4]

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} + \frac{\partial F}{\partial \dot{q}_i} = Q_i \qquad (1)$$

where $Q_i$ is an externally applied force and $F$ is a dissipation function to be considered later. For a fluid region (or element) of volume $V$ we have

$$L = \tfrac{1}{2}\int_{\text{vol}} \rho \dot{\mathbf{f}}^T \dot{\mathbf{f}}\, dV - \tfrac{1}{2}\int_{\partial V} g\rho w^2 \hat{\mathbf{k}}\cdot\hat{\mathbf{n}}\, dA - \tfrac{1}{2}\int_{\text{vol}} K\varepsilon_V{}^2\, dV - \tfrac{1}{2}\int_{\text{surface}} S[(w_{,x})^2 + (w_{,y})^2]\, dA \qquad (2)$$

where a comma denotes partial differentiation. The four integrals in Eq. (2) are explained as follows. The first represents kinetic energy. The second represents the potential of fluid elevated by sloshing, and is explained by noting that vertical displacement $w$ of the element boundaries causes infinitesimal weights $g\rho w\, dx\, dy$ to have their mass centers raised an amount $w/2$. In this expression $dx\, dy$ is the horizontal projection $\hat{\mathbf{k}}\cdot\hat{\mathbf{n}}\, dA$ of an area $dA$ of the element boundary. The same force $g\rho w\, dx\, dy$ is produced by a Winkler foundation of modulus $\rho g$, thus the integral might be viewed as energy stored by an elastic foundation. The third integral in Eq. (2) represents energy stored by volume change.[4] The last integral represents the potential of surface tension, derived by considering the constant force $S$ acting through distances produced by surface waves. For example, with $dw/dx$ small the $x$ direction distance is

$$\int_0^l \left[1 + \left(\frac{dw}{dx}\right)^2\right]^{1/2} dx - l \approx \int_0^l \frac{1}{2}\left(\frac{dw}{dx}\right)^2 dx \qquad (3)$$

With definitions given in the Nomenclature, Eqs. (1) and (2) yield

$$\int_{\text{vol}} \rho \mathbf{N}^T\mathbf{N}\, dV \ddot{\mathbf{q}} + \left(\int_{\text{surface}} S(\mathbf{N}_{3,x}{}^T\mathbf{N}_{3,x} + \mathbf{N}_{3,y}{}^T\mathbf{N}_{3,y})\, dA + \int_{\partial V} g\rho \mathbf{N}_3{}^T\mathbf{N}_3 \hat{\mathbf{k}}\cdot\hat{\mathbf{n}}\, dA + \int_{\text{vol}} \mathbf{D}^T K \mathbf{D}\, dV\right)\mathbf{q} = \mathbf{Q} \qquad (4)$$

The coefficient of $\ddot{\mathbf{q}}$ is the familiar mass matrix.[3] The integral involving $S$ is a surface tension stiffness matrix; in form it resembles an initial stress or geometric stiffness matrix.[3] The integral involving $\rho g$ is the slosh stiffness matrix of Ref. 2. If $\rho$ is constant the contributions from adjacent elements cancel at submerged nodes and the slosh stiffness matrix couples only the surface nodes. In other words, vertical displacement of submerged fluid contributes nothing to slosh stiffness if it is merely replaced by other fluid of the same density. The final integral in Eq. (4) is the compressibility stiffness matrix. If the fluid is incompressible the constraint equation

$$\varepsilon_V = \int_{\text{vol}} \mathbf{Dq}\, dV = 0 \qquad (5)$$

is written for each element, thus eliminating many degrees of freedom.[2]

Viscous effects were not considered by Hunt[1,2] but may be included as follows. The dissipation function is[4]

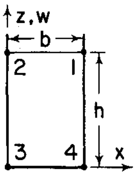


Fig. 1 One-element model for sample fluid sloshing problem.

$$F = \tfrac{1}{2}\int_{\text{vol}} \dot{\boldsymbol{\varepsilon}}^T \mathbf{C}\dot{\boldsymbol{\varepsilon}}\, dV = \tfrac{1}{2}\dot{\mathbf{q}}^T \int_{\text{vol}} \mathbf{B}^T\mathbf{C}\mathbf{B}\, dV \dot{\mathbf{q}} \tag{6}$$

where **C** is a symmetric matrix with nonzero terms $C_{11} = C_{22} = C_{33} = 4\mu/3$, $C_{12} = C_{13} = C_{23} = -2\mu/3$ and $C_{44} = C_{55} = C_{66} = \mu$. Thus, according to Eq. (1), the integral in Eq. (6) is a damping matrix and appears on the left hand side of Eq. (4). The same result is obtained if the left-hand side of Eq. (4) is augmented by the initial-stress matrix[3]

$$\int_{\text{vol}} \mathbf{B}^T\boldsymbol{\sigma}\, dV = \int_{\text{vol}} \mathbf{B}^T\mathbf{C}\dot{\boldsymbol{\varepsilon}}\, dV = \int_{\text{vol}} \mathbf{B}^T\mathbf{C}\mathbf{B}\, dV\dot{\mathbf{q}} \tag{7}$$

As an example, consider plane motion in a rectangular tank of fluid, Fig. 1. Let the thickness be one unit and $\rho$ constant. The fluid is modeled by a single linear element having four corner nodes.[3] The only nonzero nodal freedoms are $w_1$ and $w_2$, hence the displacement field is

$$w = (z/h)[(x/b)w_1 + (1-x/b)w_2] \tag{8}$$

Equations (4, 6, and 8) yield

$$\frac{\rho hb}{18}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\begin{Bmatrix}\ddot{w}_1\\ \ddot{w}_2\end{Bmatrix} + \frac{\mu}{9hb}\begin{bmatrix}4b^2+3h^2 & 2b^2-3h^2\\ 2b^2-3h^2 & 4b^2+3h^2\end{bmatrix}\begin{Bmatrix}\dot{w}_1\\ \dot{w}_2\end{Bmatrix} + \left(\frac{S}{b}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} + \left(\frac{\rho g b}{6} + \frac{Kb}{6h}\right)\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\right)\begin{Bmatrix}w_1\\ w_2\end{Bmatrix} = \mathbf{Q} \tag{9}$$

For harmonic motion of an incompressible fluid, $w_1 = -w_2 = a\sin\omega t$ and $\ddot{w}_1 = -\ddot{w}_2 = -a\omega^2\sin\omega t$. Let us set $\mu = S = \mathbf{Q} = 0$ and $h = 3b/2$. The solution of the eigenvalue problem is $\omega^2 = 3.00g/h$. For the same problem, using six elements of a different type, Ref. 2 obtains $\omega^2 = 3.46g/h$. The exact solution, Eq. (10) of Ref. 2, is $\omega^2 = 4.71g/h$.

A better fit to actual surface waves is achieved by use of the same cubic as used for beam elements. Linear edge displacements may be retained for submerged element boundaries.

## References

[1] Hunt, D. A., "Discrete Element Idealization of an Incompressible Liquid for Vibration Analysis," *AIAA Journal*, Vol. 8, No. 6, June 1970, pp. 1001–1004.

[2] Hunt, D. A., "Discrete Element Structural Theory of Fluids," *AIAA Journal*, Vol. 9, No. 3, March 1971, pp. 457–461.

[3] Zienkiewicz, O. C., *The Finite Element Method in Engineering Science*, McGraw-Hill, London, 1971, pp. 16–24, 108, 326–327, 421–423.

[4] Rayleigh, J. W. S., *Theory of Sound*, Dover, New York, 1945, Vol. I, pp. 102–103, Vol. II, pp. 312–315.

# Errata

## Eigenvalues and Eigenvectors for Solutions to the Radiative Transport Equation

J. A. Roux,* D. C. Todd,† and A. M. Smith‡
*ARO Inc., Arnold Air Force Station, Tenn.*

[AIAA J. **10**, 973–976 (1972)]

EQUATION (23) should read

$$I(\tau, x_i) = c_1 + c_p(\tau - x_i) + \sum_{j=2}^{p/2}\left\{\frac{1-\lambda_j x_j}{1-\lambda_j^2 x_i^2}\right\} \times \{c_j(1-\lambda_j x_i)e^{\lambda_j\tau} + c_{p+1-j}(1+\lambda_j x_i)e^{-\lambda_j\tau}\} \tag{1}$$

instead of

$$I(\tau, x_i) = c_1 + c_p\tau + \sum_{j=2}^{p/2}\left\{\frac{1-\lambda_j x_j}{1-\lambda_j^2 x_i^2}\right\} \times \{c_j(1-\lambda_j x_i)e^{\lambda_j\tau} + c_{p+1-j}(1+\lambda_j x_i)e^{-\lambda_j\tau}\} \tag{2}$$

The reason for this modification is because of the change in the form of the eigenvectors associated with the repeated eigenvalue $\lambda_1 = \lambda_p = 0$. The general solution of Eq. (2), p. 974, is given by Eq. (21). However, in order to provide two linearly independent solutions for the repeated eigenvalue, Eq. (21) becomes

$$I(\tau, x_i) = (c_1 + c_p\tau)v_{i1}e^{\lambda_1\tau} + c_p v_{ip}e^{\lambda_p\tau} + \sum_{j=2}^{p-1} c_j v_{ij}e^{\lambda_j\tau}, \quad i = 1,\ldots,p \tag{3}$$

where the exponentials outside of the summation are equal to unity since $\lambda_1 = \lambda_p = 0$. The constants $c_1$ and $c_p$ are integration constants and $v_{i1}$ and $v_{ip}$ are the $i$th components of the first and $p$th eigenvectors. Now for $\lambda_1 = 0$, Eq. (20) is valid for determining $v_{i1}$, which yields

$$v_{i1} = [(1-\lambda_1 x_1)/(1+\lambda_1 x_i)]v_{p1}, \quad i = 1,\ldots,p \tag{4}$$

Since (as shown before) $v_{p1}$ is arbitrary, choose $v_{p1} = 1$ and then because $\lambda_1 = 0$, Eq. (4) yields

$$v_{i1} = 1, \quad i = 1,\ldots,p \tag{5}$$

Now to determine the second "independent" eigenvector associated with the repeated eigenvalue, a special procedure[1] must be used. The eigenvector for the repeated eigenvalue ($\lambda_p = 0$) must satisfy

$$\sum_{j=1}^{p}(B_{ij} - \delta_{ij}\lambda_p)v_{jp} = v_{i1} \tag{6}$$

Note $v_{i1}$ is determined by a similar expression except the right-hand side of Eq. (6) would be zero and subscript $p$ would be replaced by subscript 1. Observing that $v_{i1} = 1$, substituting for $B_{ij}$ ($W = 1.0$) from Eq. (3), p. 974, and simplifying the algebra, Eq. (6) becomes

$$\tfrac{1}{2}\sum_{j=1}^{p} a_j v_{jp} = v_{ip}(\lambda_p x_i + 1) + x_i, \quad i = 1,\ldots,p \tag{7}$$

Now for the $p$th eigenvector, subtract the $(p+1-k)$th equation from the other $p-1$ equations. Then all the equations except the $(p+1-k)$th equation become

$$0 = [v_{ip}(\lambda_p x_i + 1) + x_i] - [(1+\lambda_p x_{p+1-k})v_{p+1-k,p} + x_{p+1-k}] \tag{8}$$

but since $\lambda_p = 0$, this yields

$$v_{ip} = v_{p+1-k,p} + x_{p+1-k} - x_i, \quad i = 1,\ldots,p \tag{9}$$

Received September 25, 1972. This research was sponsored by the Arnold Engineering Development Center, Air Force Systems Command, under Contract F40600-69-C-0001 with ARO Inc.

Index categories: Radiation and Radiative Heat Transfer; Atmospheric, Space, and Oceanographic Sciences.